ON A MINIMUM PRINCIPLE IN DYNAMICS OF RIGID-PLASTIC BODIES

(OB ODNOM MINIMAL'NOM PRINTSIPE V DINAMIKE ZHESTKO-PLASTICHESKOGO TELA)

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Uniqueness of the solution and variational principles are well known for the statics of rigid-plastic bodies [1]. The situation is completely different for the dynamics of rigid-plastic bodies. The first and only work on the variational methods in this area is a paper by Rzhanitsyn [2], where the author proposes to use Lagrange's principle for the determination of motion of beams and plates beyond the elastic range. A rigid condition however is imposed in that the form of the motion remains unchanged in time. Moreover, the question whether the Lagrange's principle is applicable to rigid-plastic bodies remains open.

Below an extremal property of the dynamics of rigid ideally plastic bodies is demonstrated. It is shown that a true instantaneous acceleration minimizes some functional, whereby the true instantaneous acceleration field is unique. This minimum principle can be used for approximate solutions of problems of dynamics of rigid-plastic bodies.

1. Let a rigid-plastic body have volume V and piecewise smooth boundary S. On a part of the surface S_T surface loads T_i are given and on the remaining part of the surface S_v velocities v_i are prescribed. At time $t = t_0$ the velocity field $v_i^*(x, y, z, t_0)$ is given in the body, i.e. the velocities of deformations $\varepsilon_{ij}^* = 1/2(v_{i,j}^* + v_{j,i}^*)$ are known. The asterisks denote here and in the sequel the true fields. The dots above the letters denote differentiation with respect to time. We neglect the changes of the geometry of the body in the process of deformation.

Admissible velocities $v_i(x, y, z, t)$ and corresponding admissible deformation velocities $\varepsilon_{ij}(x, y, z, t) = 1/2(v_{i,j} + v_{j,i})$ will satisfy the following:

1. they satisfy kinematic boundary conditions on S_v;

2. they do not violate continuity of the body;
3. they satisfy the condition of incompressibility of the material, $v_{i,i} = 0$;
4. the deformation velocities satisfy the flow law associated with the yield function $f(\sigma_{ij})$;
5. at the instant $t = t_0$ admissible velocities coincide with true velocities

$$v_i(x, y, z, t)_{t=t_0} = v_i^*(x, y, z, t_0)$$

A kinematically admissible stress field σ_{ij} consists of stresses related by the flow law with $\dot{\varepsilon}_{ij}$. If $\dot{\varepsilon}_{ij} = 0$ then any arbitrary state of stress is possible either inside or on the boundary of the yield surface. If $\dot{\varepsilon}_{ij} \neq 0$ then σ_{ij} is represented by a point on the yield surface, whose outward normal coincides with $\dot{\varepsilon}_{ij}$. Various possible cases are shown in Fig. 1.

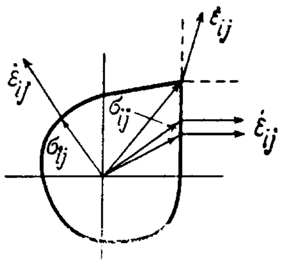

Fig. 1.

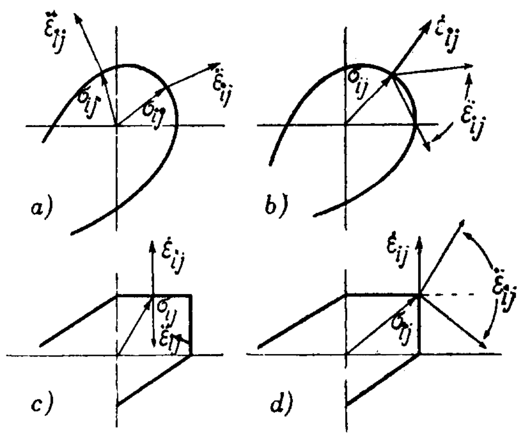

Fig. 2.

For strictly convex yield surfaces the stresses σ_{ij} are determined uniquely. An admissible acceleration field is obtained by differentiation of an admissible velocity field

$$w_i(x, y, z, t_0) = \left(\frac{\partial v_i(x, y, z, t)}{\partial t}\right)_{t=t_0}, \qquad \ddot{\varepsilon}_{ij} = \frac{1}{2}(w_{i,j} + w_{j,i}) \tag{1.1}$$

The true accelerations w_i^* and $\ddot{\varepsilon}_{ij}^*$ are distinguished from all possible ones by satisfying the equations of motion

$$\sigma^*_{ij,j} + p_i - mw_i^* = 0 \tag{1.2}$$

where p_i are inertia forces, m is the density, σ_{ij} are true stresses, and therefore they satisfy the boundary conditions on S_T

$$\sigma^*_{ij} n_j = T_i \tag{1.3}$$

It is necessary now to consider the limitations placed on the admissible deformation velocities by the associated flow law.

Let $f(\sigma_{ij}) = 0$ be a convex, smooth surface; then

$$\dot{\varepsilon}_{ij} = \lambda \frac{\partial f}{\partial \sigma_{ij}} \qquad \begin{cases} \lambda = 0, & \text{if } f < 0 \\ \lambda \geqslant 0, & \text{if } f = 0 \end{cases} \tag{1.4}$$

where λ and $\partial f / \partial \sigma_{ij}$ are functions of time and space coordinates. Differentiating it with respect to time, we have

$$\ddot{\varepsilon}_{ij} = \dot{\lambda} \frac{\partial f}{\partial \sigma_{ij}} + \lambda \frac{\partial}{\partial t} \frac{\partial f}{\partial \sigma_{ij}} \qquad \begin{cases} \dot{\lambda} = 0, \ \lambda = 0, & \text{if } f < 0 \\ \dot{\lambda} \geqslant 0, & \text{if } f = 0, \ \lambda = 0 \\ \dot{\lambda} \text{ arbitrary} & \text{if } \lambda > 0 \end{cases} \tag{1.5}$$

Thus, for $\lambda = 0$ the deformation accelerations $\varepsilon_{ij} = \lambda \, \partial f / \partial \sigma_{ij}$ satisfy the associated flow law with a nonnegative factor λ (Fig. 2*a*).

If $\lambda > 0$, then no limitations are set for ε_{ij} (Fig. 2*b*).

Let us now consider a piecewise smooth yield surface

$$f_k = a_{ij_k} \sigma_{ij}$$

Then

$$\dot{\varepsilon}_{ij} = \lambda_k a_{ij_k} \qquad \begin{cases} \lambda_k = 0, & \text{if } f_k < 0 \\ \lambda_k \geqslant 0, & \text{if } f_k = 0 \end{cases}$$

It follows that

$$\ddot{\varepsilon}_{ij} = \dot{\lambda}_k a_{ij_k} \begin{cases} \dot{\lambda}_k = 0, & \text{if } f_k < 0 \\ \dot{\lambda}_k \geqslant 0, & \text{if } f_k = 0, \ \lambda_k = 0 \\ \dot{\lambda}_k \text{ arbitrary} & \text{if } \lambda_k > 0 \end{cases} \tag{1.6}$$

Consequently, for $\varepsilon_{ij} = 0$ the deformation acceleration satisfies the generalized associated flow law.

If

$$\lambda_r \neq 0, \qquad \lambda_k = 0, \qquad f_k < 0 \ . \ (k \neq r)$$

then ε_{ij} is orthogonal to the surface $f_r = a_{ij_r}\sigma_{ij}$ and can be directed along the inward normal (Fig. 2*c*).

If

$$\lambda_r > 0, \quad \lambda_k = 0 \quad (k \neq r), \quad f_k < 0 \quad (k \neq r, \ r+1), \quad f_{r+1} = 0$$

then the deformation accelerations are directed as shown in Fig. 2*d*.

If the stresses correspond to the corner points, then no limitations are set on ε_{ij}.

2. *Theorem.* From all kinematically admissible ε_{ij}, w_i and σ_{ij}, the true ε_{ij}^*, w_i^* and σ_{ij}^* at each instant are those which minimize the expression

$$J = \int_V \frac{mw_i^2}{2}\, dV - \int_V p_i w_i dV - \int_{S_T} T_i w_i ds + \int_V \sigma_{ij}\ddot{\varepsilon}_{ij} dV \tag{2.1}$$

Proof. It is necessary to prove that

$$J^* - J = \int_V \frac{m}{2}(w_i^{*2} - w_i^2)\, dV - \int_V p_i (w_i^* - w_i)\, dV -$$

$$- \int_S T_i (w_i^* - w_i)\, dS + \int_V (\sigma_{ij}^* \ddot{\varepsilon}_{ij}^* - \sigma_{ij}\ddot{\varepsilon}_{ij})\, dV \leqslant 0 \tag{2.2}$$

Let us transform the nonpositive part in (2.2)

$$\int_V \frac{m}{2}(w_i^{*2} - w_i^2)\, dV =$$

$$= \int_V m w_i^* (w_i^* - w_i)\, dV - \int_V \frac{m (w_i^* - w_i)^2}{2}\, dV \tag{2.3}$$

Using (1.2) and (1.3), we transform the following integrals

$$\int_V m w_i^* (w_i^* - w_i)\, dV - \int_V p_i (w_i^* - w_i)\, dV - \int_S T_i (w_i^* - w_i)\, ds =$$

$$= \int_V (m w_i^* - p_i)(w_i^* - w_i)\, dV - \int_S \sigma_{ij}^* n_j (w_i^* - w_i)\, ds = \tag{2.4}$$

$$= \int_V \sigma_{ij,j}^* (w_i^* - w_i)\, dV - \int_V [\sigma_{ij}^* (w_i^* - w_i)_j]\, dV =$$

$$= -\int_V \sigma_{ij}^* [w_{i,j}^* - w_{i,j}]\, dV = -\int_V \sigma_{ij}^* (\ddot{\varepsilon}_{ij}^* - \ddot{\varepsilon}_{ij})\, dV$$

Here the symmetry of σ_{ij}^* was used, and it was assumed that σ_{ij}^*, w_i^* and w_i have continuous partial derivatives with respect to the space coordinates.

Substituting (2.4) and (2.3) into (2.2) we obtain

$$* - J = -\int_V \frac{m (w_i^* - w_i)^2}{2}\, dV + \int_V \ddot{\varepsilon}_{ij} (\sigma_{ij}^* - \sigma_{ij})\, dV \qquad (2.5)$$

Let us investigate the expression

$$\ddot{\varepsilon}_{ij} (\sigma_{ij}^* - \sigma_{ij}) \qquad (2.6)$$

If at the instant $t = t_0$ we have $\varepsilon_{ij}^* = 0$, $f(\sigma_{ij}) < 0$, then $\ddot{\varepsilon}_{ij} = 0$.

If $\dot{\varepsilon}_{ij}^* \neq 0$ and $f(\sigma_{ij}) = 0$ represents a smooth, convex surface, then σ_{ij} are determined uniquely from (1.4), and kinematically admissible stresses coincide with σ_{ij}^*, the Expression (2.6) at this point reduces to zero.

If $f(\sigma_{ij}) = 0$ and $\dot{\varepsilon}_{ij}^* = 0$, then because of (1.5) we have $\ddot{\varepsilon}_{ij}(\sigma_{ij}^* - \sigma_{ij}) \leqslant 0$, since this quantity is proportional to the scalar product of the external normal to the surface f at the point σ_{ij} by the vector directed from σ_{ij} to σ_{ij}^*. If $\dot{\varepsilon}_{ij}^*$ defines a hyperplane f_k on the yield surface, then σ_{ij}^* and σ_{ij} are both in this hyperplane. This is so because due to (1.7) the expression (2.6) is nonpositive.

Consequently, for all cases (2.6) and thus (2.5) are nonpositive. The theorem thus is proved completely.

From this theorem follows immediately the uniqueness of the acceleration field at each instant. Suppose there are two complete solutions $\ddot{\varepsilon}_{ij1}$; w_{i1}; σ_{ij1} and $\ddot{\varepsilon}_{ij2}$; w_{i2}; and σ_{ij2}. Let the functionals J_1 and J_2 correspond to these solutions. From the theorem it follows that $J_1 < J$ for an arbitrary admissible J, including J_2. But by the same token J_2 is smaller than arbitrary J including J_1, thus $J_1 - J_2 = 0$. This expression may be represented in the form analogous to (2.5)

$$J_1 - J_2 = -\int_V \frac{m (w_{i1} - w_{i2})^2}{2}\, dV + \int_V \ddot{\varepsilon}_{ij1} (\sigma_{ij2} - \sigma_{ij1})\, dV = 0$$

Since both integrals are nonpositive, each equals zero. It follows from here that $w_{i1} \equiv w_{i2}$, in the whole volume V, and consequently $\ddot{\varepsilon}_{ij1} = \ddot{\varepsilon}_{ij2}$. The stresses σ_{ij} at the points where $\dot{\varepsilon}_{ij} = 0, \ddot{\varepsilon}_{ij} = 0$ remain undetermined. If $\dot{\varepsilon}_{ij} \neq 0$ or $\dot{\varepsilon}_{ij} = 0$, $\ddot{\varepsilon}_{ij} \neq 0$, then σ_{ij} is determined from the mechanism of flow, and in the case of a piecewise linear yield surface can be multivalued.

3. It is possible to generalize the minimum principle proved above to the acceleration fields which have discontinuities on the surfaces dividing the body into a finite number of regions inside which the accelerations are continuous. Such a generalization is necessary since in practical problems the accelerations, as a rule, are discontinuous. To the Expressions (2.1), representing functionals J, one must add terms corresponding to the work of the stresses causing accelerations on the discontinuity surfaces, i.e.

$$J = \int_V \frac{m}{2} w_i^2 dV - \int_V p_i w_i \, dV - \int_{S_T} T_i w_i dS + \int_V \sigma_{ij} \ddot{\varepsilon}_{ij} dV + \int_l \sigma_{ij} n_j [w^i] \, dl \tag{3.1}$$

whereby the last integral is extended over all discontinuity surfaces l, where the jump of the accelerations $[w_i] \neq 0$; here and in the sequel the brackets denote the magnitudes of the jumps.

In (2.2) additional terms are

$$\int_{l^*} \sigma_{ij}^* n_j^* [w_i^*] \, dl^* - \int_l \sigma_{ij} n_j [w_i] \, dl$$

where l^* is the discontinuity surface of the true accelerations, and l^* generally does not coincide with l.

In (2.4) the surface integral is transformed as follows

$$-\int_S T_i (w_i^* - w_i) \, dS = -\int_V \sigma_{ij}^* \ddot{\varepsilon}_{ij}^* \, dV + \int_V \sigma_{ij}^* \ddot{\varepsilon}_{ij} dV -$$

$$-\int_{l^*} \sigma_{ij}^* n^* [w_i^* \;] \, dl^* + \int_l \sigma_{ij}^* n_j [w_i] \, dl$$

In the final expression for (2.5) there will appear an integral

$$\int_l (\sigma_{ij}^* - \sigma_{ij}) \, n_j \, [w_i] \, dl \tag{3.2}$$

which can be estimated as being nonpositive. To do this we have to investigate the conditions at the jumps of the accelerations in a continuous body.

Consider a case of plane deformation. The plasticity condition and the flow law are the usual ones

$$\frac{(\sigma_x - \sigma_y)^2}{4} + \tau_{xy}^2 = k^2 \tag{3.3}$$

$$\dot{\varepsilon}_{xx} = -\dot{\varepsilon}_{yy} = \lambda \frac{\sigma_x - \sigma_y}{2}, \qquad \dot{\varepsilon}_{xy} = 2\lambda\tau_{xy} \tag{3.4}$$

Let point P belong to l for $t = t_0$. Let us introduce for simplicity a local system of coordinate axes with the origin at P and the x-axis directed normally to the discontinuity line.

If $[\varphi] = 0$ on l, then the following conditions are satisfied

$$\left[\frac{\partial\varphi}{\partial t}\right] + G\left[\frac{\partial\varphi}{\partial x}\right] = 0, \qquad \left[\frac{\partial\varphi}{\partial y}\right] = 0 \tag{3.5}$$

where G is the speed of the discontinuity line. From the condition of conservation of mass follows that $[v_x] = 0$ in the incompressible material along the moving discontinuity line.** We have therefore from (3.5)

$$[w_x] + G\left[\frac{\partial v_x}{\partial x}\right] = 0$$

Thus $[w_x] \neq 0$ only for $G \neq 0$ and $[\partial v_x/\partial x] \neq 0$, which, because of the incompressibility and kinematic conditions (3.5), is possible only for $[v_y] \neq 0$ for $t = t_0$. The same conclusions follow in a more general form from the Thomas' book [3]. Thus, if the discontinuity line is stationary, or if the velocities do not suffer there a strong discontinuity, then $[w_x] = 0$ and the integrand in (3.2) is

$$D = (\tau_{xy}^* - \tau_{xy})\,[w_y] \tag{3.6}$$

If $\lambda = 0$, then ε_{xx}, ε_{yy}, ε_{xy} satisfy (3.4) with the multiplier $\lambda \geqslant 0$. If we consider the discontinuity line as a narrow strip with rapid, but continuous variations of w_y, then it easily follows from (3.3) and (3.4) that $\tau_{xy} = \pm k$ and the sign here coincides with the sign of $[w_y]$. Moreover, since $|\tau_{xy}^*| \leqslant k$, then $D \leqslant 0$.

If $\lambda > 0$, then a kinematically admissible τ_{xy} is determined from (3.4) uniquely, and consequently

** Exceptional cases constitute linear problems where a velocity jump can occur because of the variation of the cross-section.

$$\tau^*_{xy} - \tau_{xy} = 0, \quad D = 0$$

Finally, $[w_x] \neq 0$ can be true only along the line of strong velocity discontinuity, thus

$$\tau_{xy} = \tau^*_{xy} = \pm k$$

and the integrand in (3.2) is

$$D = [w_x](\sigma_x{}^* - \sigma_x) \tag{3.7}$$

Considering the discontinuity line as a limiting case of a strip of width Δx, where w_x is varying rapidly but continuously, and because of the incompressibility of the material, we can write $\Delta w_x/\Delta x + \Delta w_y/\Delta y = 0$, and letting Δx and Δy tend to zero, we see that w_y suffers a discontinuity at each point in the direction tangential along the whole discontinuity line, which seems to be impossible. Assuming, however, that this is true, we have to take into account the work of the stresses done on these velocity discontinuities, which again leads to $J^* \leqslant J$.

4. For bending of plates it is convenient to use generalized variables: bending and twisting moments M_x, M_y and T, and the rate of change of the curvatures and the twist

$$\dot{\varkappa}_x = -\frac{\partial^2 v}{\partial x^2}, \quad \dot{\varkappa}_y = -\frac{\partial^2 v}{\partial y^2}, \quad \dot{\tau} = -\frac{\partial^2 v}{\partial x \partial y},$$

where v is the velocity of bending of a plate.

The von Mises yield condition and the flow law are known:

$$M_x{}^2 + M_y{}^2 - M_x M_y + 3T^2 = \frac{3}{4} M_0{}^2 \tag{4.1}$$

$$\dot{\varkappa}_x = \lambda(2M_x - M_y), \quad \dot{\varkappa}_y = \lambda(2M_y - M_x), \quad \dot{\tau} = 6\lambda T \tag{4.2}$$

Acceleration of the dissipation per unit length is expressed as $[-\partial w/\partial x]M_x + [-\partial w/\partial y]T$. In order to prove the minimum property of J for the discontinuous velocities, it is necessary to determine the sign of

$$D = \left[-\frac{\partial w}{\partial x}\right](M_x{}^* - M_x) + \left[-\frac{\partial w}{\partial y}\right](T^* - T) \tag{4.3}$$

where w is the acceleration of the deflection.

If the discontinuity line coincides with the region of the plastic hinge then

$$[\partial v / \partial x] \neq 0, \quad M_x = M_x{}^* = \pm M_0, \quad T = T^* = 0 \tag{4.4}$$

and consequently $D = 0$.

If $[\partial v/\partial x] = 0$ but $[\partial^2 v/\partial x^2] \neq 0$ then (4.4) is still true [4]. If $\dot{\kappa}_x$, $\dot{\kappa}_y$, $\dot{\tau}$ are continuous for $t = t_0$, but different from zero, then $D = 0$, since (4.2) determine kinematically admissible M_x, M_y, T uniquely.

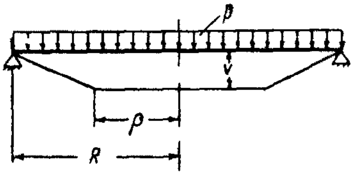

Fig. 3.

Finally, let

$$\frac{\partial^2 v}{\partial x^2} = 0, \qquad \frac{\partial^2 v}{\partial y^2} = 0, \qquad \frac{\partial^2 v}{\partial x \partial y} = 0$$
$$\text{for } t = t_0$$

From the continuity condition for v along an arbitrary discontinuity line $[\partial v/\partial y] \equiv 0$, and thus

$$\left[\frac{\partial w}{\partial y}\right] + G\left[\frac{\partial^2 v}{\partial x \partial y}\right] = 0$$

and hence in this case

$$\left[\frac{\partial w}{\partial y}\right] = 0 \qquad \text{for } t = t_0$$

If in addition $[\partial w/\partial x] \neq 0$ then, taking into account that $\ddot{\kappa}_x$, $\ddot{\kappa}_y$, $\ddot{\tau}$ satisfy the flow law then, in an analogous manner as for the case of strong discontinuous velocities, we get $M_x = \pm M_0$, but $|M_x{}^*| \leq M_0$ and $D \leq 0$.

Applying Hopkins results [5], we can consider a plate using the Tresca plasticity condition. In particular, it is necessary to consider that along the discontinuity line the bending moment equals to the limiting value.

The minimum principle obtained above is easily applied to an approximate solution of the problem.

As an example let us consider the well known problem solved by Hopkins and Prager [6].

A circular, simply supported plate is subjected to uniformly distributed load p during $0 \leq t < t_1$, and then the load is removed. (Tresca yield condition.)

We shall not vary the functional J directly, but consider that the distribution of velocities at every instant has the form of a truncated cone, Fig. 3

$$\dot{y}=0, \qquad \ddot{y}=0 \quad \text{for } r=R$$

$$\dot{y}(r)=0, \qquad \ddot{y}(r)=\lim_{\Delta t\to 0}\frac{\dot{y}(r,\Delta t)}{\Delta t} \quad \text{for } t=0 \tag{4.5}$$

Thus, at the initial instant, the distribution of y coincides with an assumed admissible distribution of velocities

$$\ddot{y}_{t=0}=\dot{v} \quad (0\leqslant r\leqslant \rho), \qquad \ddot{y}_{t=0}=\frac{r-R}{\rho-R}\dot{v} \quad (\rho\leqslant r\leqslant R) \tag{4.6}$$

For the functional we have

$$J=2\pi\left[\frac{m}{2}\int_0^\rho \dot{v}^2 r\,dr+\frac{m}{2}\int_\rho^R\left(\frac{r-R}{\rho-R}\right)^2\dot{v}^2 r\,dr-\right.$$

$$\left.-p\dot{v}\int_0^\rho r\,dr-p\dot{v}\int_\rho^R\frac{r-R}{\rho-R}r\,dr+M_0\int_\rho^R\frac{\dot{v}}{R-\rho}dr+M_0\rho\frac{\dot{v}}{R-\rho}\right]=$$

$$=2\pi\left[\frac{m\dot{v}^2}{24}(3\rho^2+2\rho R+R^2)-\frac{p\dot{v}}{6}(\rho^2+\rho R+R^2)+\frac{\dot{v}M_0R}{R-\rho}\right]$$

Equating to zero the derivatives J with respect to $\dot{v}$ and ρ, we obtain equation for $\dot{v}$ and ρ. Thus for ρ we have

$$\rho=0, \qquad p=12M_0\frac{R}{(R-\rho)^2(R+\rho)} \qquad (\rho\neq 0, \text{ of } p\geqslant 12M_0/R^2) \tag{4.7}$$

Some calculations result in

$$\dot{v}=\frac{(pR^2-6M_0)}{mR^2} \quad \text{for } \rho=0, \qquad \dot{v}=\frac{p}{m} \quad \text{for } \rho\neq 0$$

For $t > t_1$ we have

$$\dot{y}=v \quad (0\leqslant r\leqslant\rho), \quad \dot{y}=\frac{r-R}{\rho-R}v, \quad (\rho\leqslant r\leqslant R)$$

It follows from here

$$\ddot{y}=\dot{v} \quad (0\leqslant r<\rho), \quad \ddot{y}=(r-R)\frac{\dot{v}(\rho-R)-v\dot{\rho}}{(\rho-R)^2} \quad (\rho<r\leqslant R)$$

and the jump

$$\left[-\frac{\partial\ddot{y}}{\partial r}\right]=-\frac{v(\rho-R)-v\dot{\rho}}{(\rho-R)^2} \qquad (r=\rho)$$

The functional J is

$$J=\frac{m}{24}\left\{\dot{v}^2(3\rho^2+2\rho R+R^2)+\left[2v\dot{\rho}\dot{v}-\frac{(v\dot{\rho})^2}{\rho-R}\right](3\rho+R)\right\}+$$

$$+M_0\frac{R}{(R-\rho)^2}[\dot{v}(R-\rho)+v\dot{\rho}]$$

Equating to zero the derivatives with respect to $\dot{v}$ and $\dot{\rho}$ we obtain differential equations with initial conditions (4.7) and $v = pt_1/m$ which determine $v(t)$ and $\rho(t)$

either

$$\dot{v} = 0, \qquad \dot{\rho}\,(R^2 + 2R\rho - 3\rho^2) = -\frac{12M_0R}{pt_1}$$

or

$$\dot{v} = -\frac{M_0 12}{R^2 m}, \qquad \rho = 0$$

After the determination of $\rho(t)$ and $v(t)$ it is possible to find the whole remaining deformation.

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